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# Almost Disturbance Decoupling for Single-Input Single-Output Nonlinear Systems

R. MARINO, W. RESPONDEK, AND A. J. VAN DER SCHAFT

**Abstract**—The almost disturbance decoupling problem for nonlinear single-input single-output systems is addressed by using singular perturbation methods and high-gain feedback. Sufficient conditions and the explicit high-gain nonlinear state feedback in solvable cases are given. They generalize both previous almost results for linear systems and exact ones for nonlinear systems. The necessity of the conditions is discussed; in particular, an example is given where the main structural condition is not satisfied and the high-gain control designed on the basis of linear approximations fails to achieve almost disturbance decoupling for the original system.

## I. INTRODUCTION

The problem of (exact) disturbance decoupling (also called disturbance rejection) by static state feedback for linear time-invariant systems was solved in [1] and [2] in terms of geometric conditions by introducing the key concept of controlled invariant subspaces. In [3] frequency domain conditions involving relative orders of the transfer matrix are given. The geometric conditions of [1] and [2] were generalized for nonlinear systems in [4] and [5] by introducing the concept of controlled invariant distributions, a differential geometric generalization of controlled invariant subspaces. In [6] the notion of relative orders, or characteristic indexes, introduced for nonlinear systems in the study of invertibility by Hirschorn [7], has been employed in order to generalize for nonlinear systems the frequency domain conditions given in [8]. In the case in which disturbance measurements are available the conditions given in [1] and [2] for linear systems are relaxed in [8] and [9]. In [10] the differential geometric conditions given in [4] and [5] are relaxed when disturbances are measured; in [6] they are formulated using characteristic indexes.

Recently there has been a new development for linear systems due to Willems [11], who posed and solved the problem of characterizing those systems for which disturbance decoupling can be achieved approximately with an arbitrary degree of accuracy (*almost disturbance decoupling* in the terminology introduced in [11]). Necessary and sufficient conditions given in [11] involve the concept of almost controlled invariant subspaces, which are generalizations of controlled invariant subspaces. Willems points out that the problem is related to high-gain feedback design since in cases in which the problem cannot be exactly solved, increasing the accuracy of the decoupling requires increasing the gains of the linear state feedback control. On the other hand, in [12] it is shown how singular perturbation techniques can be used in the analysis of high-gain feedback systems; the reader is also referred to the recent book [13] on singular perturbation methods in control. In [14] those methods are used in order to design a high-gain feedback which almost decouples the disturbances for a restricted subclass of linear systems; see also [15] for further extensions. In [16] the necessary and sufficient conditions given in [11] are reobtained in the  $L^\infty$  case using singular perturbation techniques and without using almost invariant subspaces. In solvable cases a high-gain feedback is explicitly given which includes the one proposed in [14] in special cases. The  $L^p$  case for arbitrary  $p$  is treated directly in [17].

In this note we address the almost disturbance decoupling problem for nonlinear single input, single output systems of the following type:

$$\dot{x} = f(x) + g(x)u(t) + \sum_{k=1}^m p_k(x)w_k(t) \quad (\Sigma)$$

$$y = h(x) \quad (1)$$

where  $f, g, p_1, \dots, p_m$  are smooth vector fields on a smooth manifold  $M$ ,  $u(t): \mathbb{R}^+ \rightarrow \mathbb{R}$  is the control,  $w(t) = (w_1(t), \dots, w_m(t))^T: \mathbb{R}^+ \rightarrow \mathbb{R}^m$  is the disturbance, and  $h: M \rightarrow \mathbb{R}$  is a smooth output function. We extend the results known for linear systems following the same approach as in [16]. In fact, the basic approximation result from singular perturbation theory (Tikhonov's theorem [18], extended to infinite time intervals in [19]) is given for nonlinear systems. High gain feedback for nonlinear systems has been investigated in [20] using these methods; see also [21]. Here we obtain sufficient conditions via a constructive proof which explicitly provides the high-gain nonlinear state feedback control that solves the problem, and which generalizes the results obtained in [16] for linear systems. The main difference is that in the nonlinear case certain conditions are required on the disturbance vector fields  $p_k$  in order to avoid an interplay between the peaking, which is induced by the high-gain feedback on some submanifolds, and the nonlinearities in  $p_k$ .

## II. EXACT AND APPROXIMATE DISTURBANCE DECOUPLING

Let us recall the following definitions (e.g., [4]). Denote by  $\psi_t(x_0, t_0, u, w)$  the flow of system  $(\Sigma)$ , starting from  $x(t_0) = x_0$  and subject to the control function  $u$  and the disturbance vector function  $w = (w_1, \dots, w_m)^T$ .

**Definition 1:** In system  $(\Sigma)$  the disturbance vector  $w$  is said to be *decoupled* from the output  $y$  if for each initial state  $x(t_0) = x_0 \in M$  and each control  $u$ , the output satisfies

$$y_1(t) := h(\psi_t(x_0, t_0, u, w_1)) = y_2(t) := h(\psi_t(x_0, t_0, u, w_2)) \quad (2)$$

for every pair of disturbance vector functions  $w_1, w_2$ , and for every time  $t \geq t_0$  for which the solutions  $\psi_t(x_0, t_0, u, w_i)$ ,  $i = 1, 2$ , are defined.

**Definition 2:** The *disturbance decoupling* problem is said to be solvable for system  $(\Sigma)$  if there exists a smooth state feedback (with  $v$  the new control)

$$u = a(x) + b(x)v, \quad b(x) \neq 0 \text{ for all } x \in M \quad (3)$$

such that in the closed-loop system the disturbance  $w(t)$  is decoupled from the output  $y$  for every control  $v(t)$ .

Next we generalize the definition of almost disturbance decoupling as given by Willems [11]; see also [14], [15], to the nonlinear case.

**Definition 3:** The *almost disturbance decoupling* problem is said to be solvable for system  $(\Sigma)$  if there exists a parameterized state feedback

$$u = a(x, \epsilon) + b(x, \epsilon)v \quad (4)$$

where  $\epsilon \in (0, \epsilon^*)$ ,  $\epsilon^* > 0$ , and  $a(x, \epsilon), b(x, \epsilon)$  are smooth functions of their arguments with  $b(x, \epsilon) \neq 0$  for all  $x \in M$  and all  $\epsilon \in (0, \epsilon^*)$ , such that in the closed-loop system the disturbance  $w(t)$  is "almost" decoupled from the output  $y$ , in the following sense. Denote for  $u$  given by (4)  $\psi_t(x_0, t_0, u, w) = \psi_t(x_0, t_0, v, w, \epsilon)$ . Then for each  $x_0 \in M$ , each control  $v$ , and every disturbance  $w$ , the output  $y = h(x)$  satisfies

$$\lim_{\epsilon \rightarrow 0} |h(\psi_t(x_0, t_0, v, w, \epsilon)) - h^L(\psi_t^L(x_0^L, t_0, v, w))| = 0 \quad (5)$$

uniformly in  $t \in [t_1, T(x_0))$ , where  $t_1 > t_0$  but otherwise arbitrary, and  $T(x_0) = \infty$  or equal to the escape time of the flow  $\psi_t$  from  $x_0$  [i.e.,  $T(x_0)$  is the smallest number for which there exists  $\epsilon \in (0, \epsilon^*)$ ,  $v$ , and  $w$ , such that  $\psi_t(x_0, t_0, v, w, \epsilon)$  is not defined for  $t \geq T(x_0)$ ]. Here  $\psi_t^L$  and  $h^L$  denote the flow and the output function of a "limiting" system  $\Sigma^L$

$$\dot{x}^L = f^L(x^L) + g^L(x^L)v(t) + \sum_{k=1}^m p_k^L(x^L)w_k(t) \quad (\Sigma^L)$$

$$y^L = h^L(x^L, v) \quad (6)$$

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of the same class as  $(\Sigma)$ , with state space  $M^L$ , and with initial condition  $x_0^L$  depending on  $x_0$ , where the disturbance  $w(t)$  is decoupled from the output  $y^L$ .

**Remark:** In Willems' definition for linear systems  $x_0 = 0$  and  $v = 0$  are fixed. Then (5) reduces to  $\lim_{\epsilon \rightarrow 0} |h(\psi_t(x_0, t_0, 0, w, \epsilon))| = 0$ . Even in the linear case Definition 3 is more general than Willems' definition [16], [17].

Two characteristic indexes can be associated with system  $(\Sigma)$ . The control characteristic index  $\rho$  is defined to be the least positive integer such that

$$L_g L_f^{\rho-1} h(x) \neq 0, \quad \text{for some } x \in M. \quad (7)$$

The index  $\rho$  is set equal to  $\infty$  if  $L_g L_f^i h(x) = 0, \forall i \geq 0, \forall x \in M$ . (This implies that the control  $u$  has no influence on the output  $y$  whatsoever.) Here the notation  $L_X \varphi$  indicates directional derivative of a function  $\varphi$  along a vector field  $X$  and is defined in local coordinates  $(x_1, \dots, x_n)$  as  $\sum_{i=1}^n \partial \varphi / \partial x_i X_i$ . (N.B.: The definition of characteristic index is not entirely standard; sometimes it is defined as  $\rho - 1$  [4], [22].) We shall make *throughout* the following assumption concerning the control characteristic index.

**Assumption 1:**  $\rho < \infty$  and  $L_g L_f^{\rho-1} h(x) \neq 0$  for every  $x \in M$ .

The disturbance characteristic index  $\nu$  is defined in a similar way:  $\nu$  is the least positive integer such that

$$L_{p_k} L_f^{\nu-1} h(x) \neq 0, \quad \text{for some } k = 1, \dots, m \text{ and some } x \in M. \quad (8)$$

We set  $\nu = \infty$  if  $L_{p_k} L_f^i h(x) = 0, 1 \leq k \leq m; \forall i \geq 0, \forall x \in M$ .

**Lemma 1** (see, e.g., [20], [22], [23], [4]):  $\rho$  satisfies  $\rho \leq n (= \dim M)$ . Around any  $x_0 \in M$  there exist  $n - \rho$  functions  $\varphi_{\rho+1}(x), \dots, \varphi_n(x)$  such that:

- i)  $\langle d\varphi_i, g \rangle = 0, \rho + 1 \leq i \leq n$ ;
- ii) the functions  $h, \dots, L_f^{\rho-1} h, \varphi_{\rho+1}, \dots, \varphi_n$  form a local coordinate system.

Denote the local coordinates as determined by Lemma 1 by  $z = (z_1, \dots, z_n)$ :

$$\begin{aligned} z_1 &= h(x), z_2 = L_f h(x), \dots, z_\rho = L_f^{\rho-1} h(x) \\ z_{\rho+1} &= \varphi_{\rho+1}(x), \dots, z_n = \varphi_n(x). \end{aligned} \quad (9)$$

In such coordinates the system  $(\Sigma)$  takes the following form.

- **Case  $\nu > \rho$**  (see [4], [20], [22]):

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad 1 \leq i \leq \rho - 1 \\ \dot{z}_\rho &= L_f^\rho h + u(t) L_g L_f^{\rho-1} h \\ \dot{z}_j &= L_f z_j + \sum_{k=1}^m w_k(t) L_{p_k} z_j, \quad \rho + 1 \leq j \leq n \\ y &= z_1. \end{aligned} \quad (10)$$

- **Case  $\nu \leq \rho$ :**

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad 1 \leq i \leq \nu - 1 \\ \dot{z}_{\nu+j} &= z_{\nu+j+1} + \sum_{k=1}^m w_k(t) L_{p_k} L_f^{\nu+j-1} h, \quad 0 \leq j \leq \rho - \nu - 1 \\ \dot{z}_\rho &= L_f^\rho h + \sum_{k=1}^m w_k(t) L_{p_k} L_f^{\rho-1} h + u(t) L_g L_f^{\rho-1} h \\ \dot{z}_j &= L_f z_j + \sum_{k=1}^m w_k(t) L_{p_k} z_j, \quad \rho + 1 \leq j \leq n \\ y &= z_1. \end{aligned} \quad (11)$$

The disturbance decoupling problem for system  $(\Sigma)$  was posed and solved

independently in [4] and [5] in the general multiinput multioutput case, using the concept of a controlled invariant distribution. In the present case, [4, Theorem 3.1] readily implies that the disturbance decoupling problem is solvable if, and only if,  $L_{p_k} L_f^i h(x) = 0, 0 \leq i \leq \rho - 1, k = 1, \dots, m$ , for any  $x \in M$ . Using the definition of  $\nu$  this immediately yields the following.

**Theorem 1** [6]: The disturbance decoupling problem is solvable if, and only if,  $\nu > \rho$ . The state feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h} (-L_f^\rho h + v) \quad (12)$$

solves the problem.

Note that the control (12) substituted in (10) makes all the states affected by disturbances, i.e.,  $z_{\rho+1}, \dots, z_n$ , unobservable from the output.

The main contribution of the present note is to show that in case  $\nu \leq \rho$  (and, hence, the (exact) disturbance decoupling problem is *not* solvable) the *almost* disturbance decoupling problem can be solved, at least provided some additional conditions are satisfied.

**Theorem 2:** Consider a single-input single-output system  $(\Sigma)$  with  $\nu \leq \rho$ .

- A) Assume that for  $k = 1, \dots, m, i = \nu - 1, \dots, \rho - 1$  and for every  $x \in M$

$$d(L_{p_k} L_f^i h)(x) \in \text{span} \{dh, \dots, d(L_f^{\nu-1} h)(x)\}. \quad (13)$$

- B) Apply the state feedback [cf. (12)]

$$u = \frac{1}{L_g L_f^{\rho-1} h} (-L_f^\rho h + v') \quad (14a)$$

and assume that the feedback transformed vector fields

$$\tilde{f} = f - g \cdot \frac{1}{L_g L_f^{\rho-1} h} L_f^\rho h, \quad \tilde{g} = g \cdot \frac{1}{L_g L_f^{\rho-1} h} \quad (15)$$

are *complete*, and that  $M$  is simply connected.

- C) Assume that the disturbances  $w(t): \mathbb{R}^+ \rightarrow \mathbb{R}^m$  are  $C^1$ , and bounded with bounded derivatives.

Then the almost disturbance decoupling problem is solvable by the parametrized state feedback resulting from the state feedback (14a) together with the "high-gain part" ( $\epsilon > 0$ )

$$\begin{aligned} v' = -\frac{1}{\epsilon^{\rho-\nu+1}} [(r_{\rho-\nu} \epsilon^{\rho-\nu} L_f^{\rho-1} h + \dots + r_0 L_f^{\nu-1} h) \\ + r_0 (q_{\nu-2} L_f^{\nu-2} h + \dots + q_0 h - v)] \end{aligned} \quad (14b)$$

where the new input  $v(t)$  is  $C^1$  and bounded with bounded derivatives, and  $r_0, \dots, r_{\rho-\nu}$  and  $q_0, \dots, q_{\nu-2}$  are chosen in such a way that the polynomials

$$\begin{aligned} r(s) &= r_0 + r_1 s + \dots + r_{\rho-\nu} s^{\rho-\nu} + s^{\rho-\nu+1} \\ q(s) &= q_0 + q_1 s + \dots + q_{\nu-2} s^{\nu-2} + s^{\nu-1} \end{aligned} \quad (16)$$

are both Hurwitz. For  $\nu > 1$  the closed-loop input-output system tends, as  $\epsilon$  goes to zero, uniformly in  $t \in [t_1, T(x_0)]$  (with  $T(x_0) = \infty$  or equal to the escape time, see Definition 3) to the globally defined linear asymptotically stable system, not affected by disturbances

$$\begin{aligned} y^L &= z_1^L \\ \dot{z}_i^L &= z_{i+1}^L, \quad 1 \leq i \leq \nu - 2 \\ \dot{z}_{\nu-1}^L &= -q_0 z_1^L - \dots - q_{\nu-2} z_{\nu-1}^L + v \end{aligned} \quad (17)$$

and if  $\nu = 1$  to the *static* system  $y^L = v$ .

**Remark 1:** If  $\nu = \rho$ , then Theorem 2 prescribes the feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h} \left\{ -L_f^\rho h - \frac{r}{\epsilon} (L_f^{\rho-1} h + q_{\rho-2} L_f^{\rho-2} h + \dots + q_0 h - v) \right\} \quad (18)$$

with  $r > 0$  and  $q(s)$  Hurwitz. On the other hand, from [10], [6] we know that in this case also the disturbance decoupling problem *with disturbance measurements* is solvable; in fact, the feedback

$$u = \frac{1}{L_g L_f^{\rho-1} h} \left( -L_f^{\rho} h - \sum_{k=1}^m w_k(t) L_{p_k} L_f^{\rho-1} h + v \right) \quad (19)$$

does the job. An advantage of the control (18) over the control (19) is that we do not need to know the precise form of the disturbance vector fields  $p_k$ , but only the disturbance characteristic index  $\nu$ .

**Remark 2:** If  $(\Sigma)$  is a linear system, then conditions A and B are trivially satisfied.

Before proving Theorem 2 we recall from [19] a basic approximation result from singular perturbation theory. Consider the system ( $\epsilon > 0$ )

$$\frac{dx}{dt} = f(t, x, y, \epsilon), \quad x(0) = \xi(\epsilon), \quad x \in \mathbb{R}^n$$

( $\Sigma_\epsilon$ )

$$\epsilon \frac{dy}{dt} = g(t, x, y, \epsilon), \quad y(0) = \eta(\epsilon), \quad y \in \mathbb{R}^m.$$

We assume that  $g(t, 0, 0, 0) = 0$  for every  $t \geq 0$ . Two systems are associated with ( $\Sigma_\epsilon$ ): the “slow” reduced one

$$\frac{dx^s}{dt} = f(t, x^s, y^s, 0), \quad x(0) = \xi(0)$$

( $\Sigma_0$ )

$$0 = g(t, x^s, y^s, 0)$$

and the “fast” reduced one, or boundary layer (with stretched time  $\tau = t/\epsilon$ )

$$(BL) \quad \frac{dy^f}{d\tau} = g(0, x^s(t), y^f + y^s(t), 0), \quad y^f(0) = \eta(0) - y^s(0).$$

We now list the assumptions required on  $f$  and  $g$ .

I) The system ( $\Sigma_0$ ) has a continuous solution  $x^s(t)$ ,  $y^s(t)$  which exists for  $0 \leq t < \infty$ .

II) The functions  $f$  and  $g$  have continuous derivatives with respect to their arguments for  $(t, x, y)$  in some neighborhood of the points  $(t, x^s(t), y^s(t))$ ,  $0 \leq t < \infty$ ,  $0 \leq \epsilon \leq \epsilon_0$ . Also the initial data  $\xi(\epsilon)$ ,  $\eta(\epsilon)$  are smooth functions of  $\epsilon$  for  $0 \leq \epsilon \leq \epsilon_0$ .

III) The matrix  $\partial g / \partial y(t, x^s(t), y^s(t), 0)$  has all its eigenvalues  $\lambda(t)$  satisfying  $\text{Re } \lambda(t) \leq -\mu < 0$  for  $0 \leq t < \infty$ .

IV) The linear system  $dx/dt = (f_x(t) - f_y(t)g_y^{-1}(t)g_x(t))x$  along  $x^s(t)$ ,  $y^s(t)$  is exponentially stable.

V) The functions  $f$ ,  $g$  and their derivatives with respect to  $t, x, y, \epsilon$  are bounded for  $0 \leq t < \infty$ ,  $|x - x^s(t)| + |y - y^s(t)| \leq \Delta$ ,  $0 \leq \epsilon \leq \epsilon_0$ , for some  $\Delta$  and  $\epsilon_0 > 0$ .

**Theorem 3 [19]:** Let conditions I–V hold. Then ( $\Sigma_\epsilon$ ) has a unique solution  $x = x(t, \epsilon)$ ,  $y = y(t, \epsilon)$  on  $0 \leq t < \infty$  for sufficiently small  $x(0)$ ,  $y(0)$ ,  $\epsilon \geq 0$ ; system (BL) has unique solution  $y^f(\tau)$ , and

$$x(t, \epsilon) = x^s(t) + 0(\epsilon)$$

$$y(t, \epsilon) = y^s(t) + y^f(\tau) + 0(\epsilon)$$

where  $0(\epsilon)$  holds uniformly for  $0 \leq t < \infty$ .

**Proof of Theorem 2:** For simplicity take  $\rho > \nu > 1$ , the cases  $\rho = \nu$  or  $\nu = 1$  are analogous. Consider local coordinates  $z = (z_1, \dots, z_n)$  as established in Lemma 1, in which the system ( $\Sigma$ ) takes the form (11). Condition A implies, according to [24, Lemma 1.2], that the  $i$ th components of the vector fields  $p_k$ , for  $i = \nu, \dots, \rho$ , are of the form

$$L_{p_k} L_f^{i-1} h(x) = \gamma_i^k(h(x)), \dots, L_f^{\nu-1} h(x) = \gamma_i^k(z_1, \dots, z_\nu) \quad (20)$$

for smooth real functions  $\gamma_i^k, \dots, \gamma_\rho^k$ ,  $k = 1, \dots, m$ , on  $\mathbb{R}^n$ . Substitution

of the control (14) in (11) now yields the system

$$y = z_1 \quad i = 1, \dots, \nu - 1$$

$$\dot{z}_i = z_{i+1},$$

$$\dot{z}_{\nu+j} = z_{\nu+j+1} + \sum_{k=1}^m w_k(t) \gamma_{\nu+j}^k(z_1, \dots, z_\nu) \quad j = 0, \dots, \rho - \nu - 1$$

$$\dot{z}_\rho = -\frac{1}{\epsilon^{(\rho-\nu+1)}} [(r_{\rho-\nu} \epsilon^{\rho-\nu} z_\rho + \dots + r_0 z_\nu)$$

$$+ r_0(q_{\nu-2} z_{\nu-1} + \dots + q_0 z_1 - v)] + \sum_{k=1}^m w_k(t) \gamma_\rho^k(z_1, \dots, z_\nu) \quad (21a)$$

together with

$$\dot{z}_j = L_f z_j + \sum_{k=1}^m w_k(t) L_{p_k} z_j \quad j = \rho + 1, \dots, n. \quad (21b)$$

Hence, the last part of the dynamics, (21b) has been made unobservable from the output  $y = z_1$  (as in the exact problem). Now condition B ensures that (21a) for  $w = 0$  is a *globally* linear system. First, the distribution  $\Delta = \mathcal{E}^\perp$ , where  $\mathcal{E} = \text{span}(dh, \dots, dL_f^{\rho-1}h)$ , is regular and the quotient  $M' := M/\Delta$  admits the structure of a smooth manifold, and the dynamics ( $\Sigma$ ) passes to  $M'$  [4]. Furthermore,  $M'$  is simply connected since  $M$  is. Denote the projection  $M \rightarrow M'$  by  $\pi$ , then completeness of  $\tilde{f}$ ,  $\tilde{g}$  implies that  $\pi_* \tilde{f}$  and  $\pi_* \tilde{g}$  are complete. The local expression (11) yields that  $\pi_* \tilde{g}$ ,  $\pi_* \text{ad}_{\tilde{f}} \tilde{g}, \dots, \pi_* \text{ad}_{\tilde{f}}^{\eta-1} \tilde{g}$  are commuting and complete vector fields (see [25], [26] for details). All this implies [25], [26] that the projected dynamics  $\pi_* \tilde{\Sigma}$ , with  $\tilde{\Sigma}$  the feedback transformed dynamics (15), are globally equivalent to a linear system and that  $M' \cong \mathbb{R}^\rho$ . Furthermore, the global linearizing coordinates are simply given as  $h$ ,  $L_f h, \dots, L_f^{\rho-1} h$ . (Note that this argument is completely independent from the vector fields  $p_k$ ). If we now rescale the state variables

$$\bar{z}_i = z_i, \quad i = 1, \dots, \nu$$

$$\bar{z}_{\nu+j} = \epsilon^j z_{\nu+j}, \quad j = 1, \dots, \rho - \nu \quad (22)$$

then system (21a) becomes

$$y = \bar{z}_1$$

$$\dot{\bar{z}}_i = \bar{z}_{i+1}, \quad i = 1, \dots, \nu - 1$$

$$\epsilon \dot{\bar{z}}_{\nu+j} = \bar{z}_{\nu+j+1} + \epsilon^{j+1} \sum_{k=1}^m w_k(t) \gamma_{\nu+j}^k(\bar{z}_1, \dots, \bar{z}_\nu), \quad j = 0, \dots, \rho - \nu - 1$$

$$\epsilon \dot{\bar{z}}_\rho = -(r_{\rho-\nu} \bar{z}_\rho + \dots + r_0 \bar{z}_\nu) - r_0(q_{\nu-2} \bar{z}_{\nu-1} + \dots + q_0 \bar{z}_1 - v)$$

$$+ \epsilon^{(\rho-\nu+1)} \sum_{k=1}^m w_k(t) \gamma_\rho^k(\bar{z}_1, \dots, \bar{z}_\nu) \quad (23)$$

with initial conditions

$$\bar{z}_i(0) = z_i(0), \quad i = 1, \dots, \nu$$

$$\bar{z}_{\nu+j}(0) = \epsilon^j z_{\nu+j}(0), \quad j = 1, \dots, \rho - \nu. \quad (24)$$

System (23) with initial conditions (24) is a singularly perturbed system ( $\tilde{\Sigma}_\epsilon$ ), to which we may apply Theorem 3 if conditions I–V are satisfied. The “fast” reduced subsystem of (23) is

$$\frac{d\bar{z}_{\nu+j}^f}{d\tau} = \bar{z}_{\nu+j+1}^f, \quad j = 0, \dots, \rho - \nu - 1 \quad (25)$$

$$\frac{d\bar{z}_\rho^f}{d\tau} = -(r_{\rho-\nu} \bar{z}_\rho^f + \dots + r_0 \bar{z}_\nu^f) - r_0(q_{\nu-2} \bar{z}_{\nu-1}^f + \dots + q_0 \bar{z}_1^f - v). \quad (25)$$

The "slow" reduced subsystem is

$$\begin{aligned} \frac{d\bar{z}_i^s}{dt} &= \bar{z}_{i+1}^s \quad i = 1, \dots, v-2 \\ \frac{d\bar{z}_{v-1}^s}{dt} &= -q_{v-2}\bar{z}_{v-1}^s - \dots - q_0\bar{z}_1^s + v(t). \end{aligned} \quad (26)$$

Let us now check conditions I-V of Theorem 3. I is trivially satisfied. II is satisfied because we assume  $v$  and  $w$  to be of class  $C^1$ . V is satisfied because  $v$  and  $w$  are also assumed to be bounded with bounded derivatives. Condition III reduces to the polynomial  $r(s)$  [see (16)] to be Hurwitz. Finally, condition IV is equivalent to the "slow" subsystem (25) being asymptotically stable, i.e., the polynomial  $q(s)$  [see (16)] being Hurwitz. Hence, Theorem 3 applies, and the limiting system is the undisturbed linear system (17).

Let us now comment on the necessity of conditions A, B, C in Theorem 2. As is clear from the proof, condition A is imposed in order to ensure that the peaking in the state variables  $z_{v+1}, \dots, z_\rho$  resulting from letting  $\epsilon$  tend to zero, does not influence the  $i$ th components of the disturbance vector fields  $p_k$  for  $i = v, \dots, \rho$ , cf. (20), and therefore the "disturbance terms"  $\sum_{k=1}^m w_k(t)\gamma_{v+j}^k(z_1, \dots, z_\rho)$ ,  $j = 0, \dots, \rho - v$ , remain bounded, which is a crucial assumption in Hoppensteadt's theorem. Indeed, the following example indicates that condition A cannot be simply omitted.

*Example 1:* Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1(t) \\ \dot{x}_2 &= x_2^3 w_2(t) + u(t), \quad y = x_1. \end{aligned} \quad (27)$$

We have  $\rho = 2$ ,  $v = 1$ ,  $h = x_1$ ,  $L_f h = x_2$ ,  $L_g L_f h = 1$ ,  $L_{p_1} h = 1$ ,  $L_{p_2} h = 0$ ,  $L_{p_1} L_f h = 0$ ,  $L_{p_2} L_f h = x_2^3$ . Condition A is violated since  $d(L_{p_2} L_f h) \notin \text{sp}\{dh\}$ . If we would take a control as suggested by Theorem 2 (which in fact solves the problem for the linearized system, see Remark 2)

$$u = -\frac{1}{\epsilon^2} (\epsilon x_2 + x_1 - v) \quad (28)$$

and substitute this in the original system we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1(t) \\ \dot{x}_2 &= -\frac{1}{\epsilon^2} (x_1 + \epsilon x_2) + x_2^3 w_2(t) + \frac{1}{\epsilon^2} v(t), \quad y = x_1. \end{aligned} \quad (29)$$

Following [27] we consider the new coordinates  $z_1 = \epsilon^{-1/2} x_1$ ,  $z_2 = \epsilon^{1/2} x_2$  and the "stretched" time  $\tau = t/\epsilon$  in which the system becomes

$$\begin{aligned} \frac{dz_1}{d\tau} &= z_2 + \epsilon^{1/2} w_1(t) \\ \frac{dz_2}{d\tau} &= -z_1 - z_2 + z_2^3 w_2(t) + v(t), \quad y = \epsilon^{1/2} z_1. \end{aligned} \quad (30)$$

If we set  $v(t) = 0$  and consider the special disturbance function  $w_1(t) = 0$ ,  $w_2(t) = 1/3$  we obtain a reversed time van der Pol equation. Every trajectory starting in the region  $z_1^2 + z_2^2 > 3^2$  tends to infinity. Now recalling Definition 3, if we consider initial conditions  $(x_1^0, 0)$  and the pair of disturbances  $(0, 1/3)$ ,  $(0, 0)$  we see that the two corresponding outputs  $y_1(t)$  and  $y_2(t)$  have opposite behavior as  $\epsilon$  tends to zero:  $\lim_{t \rightarrow \infty} y_1(t) = \infty$  whereas  $\lim_{t \rightarrow \infty} y_2(t) = 0$ . In conclusion, almost disturbance decoupling has not been achieved.

However, for Hoppensteadt's theorem we do not really need that  $\gamma_i^k$ ,  $i = v, \dots, \rho$ , do not depend at all on the peaking variables  $z_{v+1}, \dots, z_\rho$ . In fact, we only need that all functions  $\gamma_i^k = L_{p_k} L_f^{i-1} h$ ,  $i = v, \dots, \rho$  remain bounded for  $\epsilon$  tending to zero. For instance, if we replace the term  $x_2^3$  in Example 1 by  $\sin x_2$ , then (28) does solve the almost disturbance decoupling problem. On the other hand, condition B is to ensure that peaking really does occur.

*Example 2:* Consider the system

$$\begin{aligned} \dot{x}_1 &= \arctg x_2 + w(t) \\ \dot{x}_2 &= u(t) \quad y = x_1. \end{aligned} \quad (31)$$

Then  $h = x_1$ ,  $L_f h = \arctg x_2$ ,  $L_f^2 h = 0$ ,  $L_g L_f h = 1/1 + x_2^2$ . Since  $\tilde{g} = (1 + x_2^2)\partial/\partial x_2$  is not complete, condition B is not satisfied. Disturbance decoupling in this case cannot be achieved to an arbitrary degree of accuracy, since the term  $\arctg x_2$  is bounded [no matter what we do with  $u(t)$ ] and so cannot overpower  $w(t)$  to any desired extent.

It is clear from the proof of Theorem 2 that we may also replace condition B by requiring that the map  $(h, L_f h, \dots, L_f^{\rho-1} h): M \rightarrow \mathbb{R}^\rho$  is a diffeomorphism (onto  $\mathbb{R}^\rho$ ). Furthermore, also completeness of  $\pi_* \tilde{f}$ ,  $\pi_* \tilde{g}$  is sufficient. Finally, condition C is imposed in order that condition V in Theorem 3 is satisfied; at least boundedness of  $w(t)$  is essential [17].

### III. CONCLUSIONS

The result stated in Theorem 2 opens a number of issues. Necessary conditions are to be found; Examples 1 and 2 seem to lead in the direction of the sufficient conditions of Theorem 2. Extensions to multiinput multioutput systems are obviously among the next steps. Actually, in case the decoupling matrix has full rank everywhere the extension is rather obvious (transform the system into decoupled form and apply for every single-input single-output system the theory of the present note). The problem of stability of the part made unobservable needs to be investigated; in particular, the effect of peaking phenomena on the unobservable dynamics is to be studied. Even though in this note a singular perturbation approach was followed, the results obtained could lead to a nonlinear generalization of almost controlled invariant subspaces introduced in [11].

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## On the Solution to the State Failure Detection Problem

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**Abstract**—This paper studies the interesting problem formulated in [1], which is to design a set of observers which can detect and locate unknown system failure to any first-order state component. This paper proposes a much simpler solution to this problem with less restrictions. Based on this solution, an extended study of the required number of observers and their orders is also provided.

### I. THE PROBLEM

Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + D(t) \\ y(t) &= Cx(t)\end{aligned}\quad (1)$$

where  $(A \in R^{n \times n}, B, C \in R^{m \times n})$  are given and observable, and  $D(t) = [D_1(t) \cdots D_n(t)]'$  represents the unknown failure of each first-order state component of system (1). The problem is to detect and identify the nonzero occurrence of any set (of size  $q < m$ ) of elements of  $D(t)$  from system input and output measurement. Because of the nature of this problem,  $D(t)$  may be completely unknown and the detection and identification should be instant and quick (not in convergence). Although much progress has been made in this particular area [5]–[13], fully meeting these requirements remains to be a challenge. Although this paper does not take noise effects into account in problem setting, the solution to this deterministic problem should provide a good basis for developing solutions to the corresponding stochastic problems.

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Recently, [1] proposed a new and interesting approach of constructing a set of  $\binom{n}{q}$  different observers of the structure

$$\begin{aligned}\dot{z}(t) &= Fz(t) + Gy(t) + TBu(t) \\ 0 &= Kz(t) + Py(t).\end{aligned}\quad (2)$$

Each of these  $\binom{n}{q}$  observers, when its output becomes nonzero, can instantly indicate the failure occurrence [or nonzero  $D(t)$ ] of system (1), and can also guarantee that this nonzero output is caused by any state failure *except* the failures of a set of  $q$  arbitrarily chosen states of system (1). Therefore, a total of  $\binom{n}{q}$  different combinations of these  $q$  states and their corresponding observers can uniquely isolate and identify which state component has actually failed (or which component of  $D(t)$  is nonzero) [1]. The constant  $q$  in [1] is fixed to be  $m - 1$ . In this paper, we will allow  $q$  to be any number between 1 and  $m - 1$ , since this gives the flexibility of choosing the total number of required observers.

In order for the observer (2) to achieve the above-stated properties, the conditions needed to be satisfied are derived in [1] and are listed in the following. Without loss of generality, we will choose the arbitrary set of  $q$  states as the first  $q$  states of system (1).

$$TA - FT = GC \quad (3)$$

$$0 = KT + PC \quad (4)$$

$$F \in R^{r \times r} \text{ is stable}$$

$$T \begin{bmatrix} I_q \\ -\frac{I_q}{0} \end{bmatrix} = 0 \quad (5)$$

and each column of

$$T \begin{bmatrix} 0 \\ -\frac{0}{I_{n-q}} \end{bmatrix} \neq 0. \quad (6)$$

The purpose of this paper is to propose a much simpler solution ( $F, T, G, K, P$ ) to conditions (3)–(6). This solution is based on the recent significant development on the solution of (3) [2]. In this solution, the restriction of [1] that the eigenvalues of  $F$  (denoted as  $s_1, \dots, s_r$ ) be identical but not equal to any eigenvalue of  $A$  is omitted. Based on this simpler solution, we have extended our study for a clear formula for the observer order  $r$  and a clear understanding of the effect of choosing different  $q$ .

### II. THE SOLUTION

**Step 0:** Find matrix  $U$  such that  $CU = [C_1: 0]_m^{n-m}$ . Compute  $\bar{A} = U'AU$ .

**Remark 0:** Matrix  $U$  is unitary and rank of  $C = m$ . Furthermore, for a unitary  $U$ ,  $\bar{A}$  can be in observable Hessenberg form (or block observable Hessenberg form if  $m > 1$ ) [2].

**Step 1:** For each  $s_i \in \{\lambda(F)\}$ , ( $i = 1, \dots, r$ ), find  $m$   $d_{ij} \in R^{1 \times n}$  vectors such that

$$d_{ij}(\bar{A} - s_i I) \begin{bmatrix} 0 \\ -\frac{0}{I_{n-m}} \end{bmatrix} = 0, \quad j = 1, \dots, m. \quad (7)$$

**Remark 1a:** There always exist  $m$  linearly independent  $d_{ij}$  vectors satisfying (7) for each  $s_i$ . Furthermore, these  $d_{ij}$  vectors are linearly independent of the rows of matrix  $C$  [2].

**Remark 1b:** Based on the Hessenberg form of  $\bar{A}$ ,  $d_{ij}$  vectors can be computed by simple backsubstitution. Furthermore, all  $d_{ij}$  vectors can be computed independently and therefore simultaneously (parallel computing) [2].

**Step 2:** Find vector  $c_i \in R^{1 \times m}$  such that

$$c_i \begin{bmatrix} d_{i1} \\ \vdots \\ d_{im} \end{bmatrix} U' \begin{bmatrix} I_q \\ -\frac{I_q}{0} \end{bmatrix} = 0, \quad i = 1, \dots, r. \quad (8)$$